2. Diversity

Main story

- Communication over a flat fading channel has poor performance due to significant probability that channel is in a deep fade.
- Reliability is increased by providing more resolvable signal paths that fade independently.
- Diversity can be provided across time, frequency and space.
- Name of the game is how to exploit the added diversity in an efficient manner.

Baseline: AWGN Channel

$$y = x + w$$

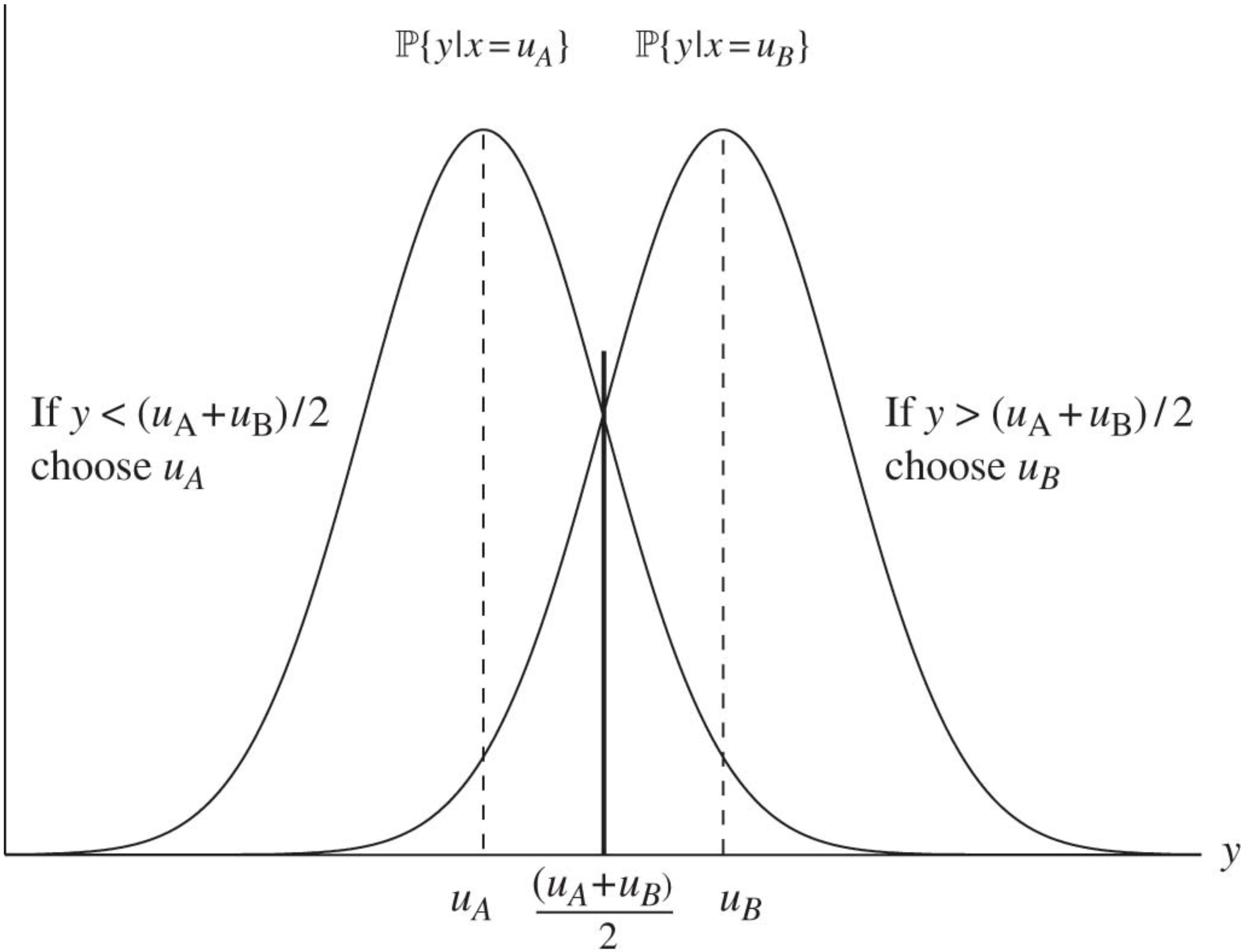
BPSK modulation $x = \pm a$

$$p_e = Q\left(\frac{a}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2\mathsf{SNR}}\right)$$

$$\mathsf{SNR} := \frac{a^2}{N_0}$$

Error probability decays exponentially with SNR.

Gaussian Detection



Rayleigh Flat Fading Channel

$$y = hx + w$$

$$h \sim \mathcal{CN}(0, 1)$$

BPSK: $x = \pm a$. Coherent detection.

Conditional on h,

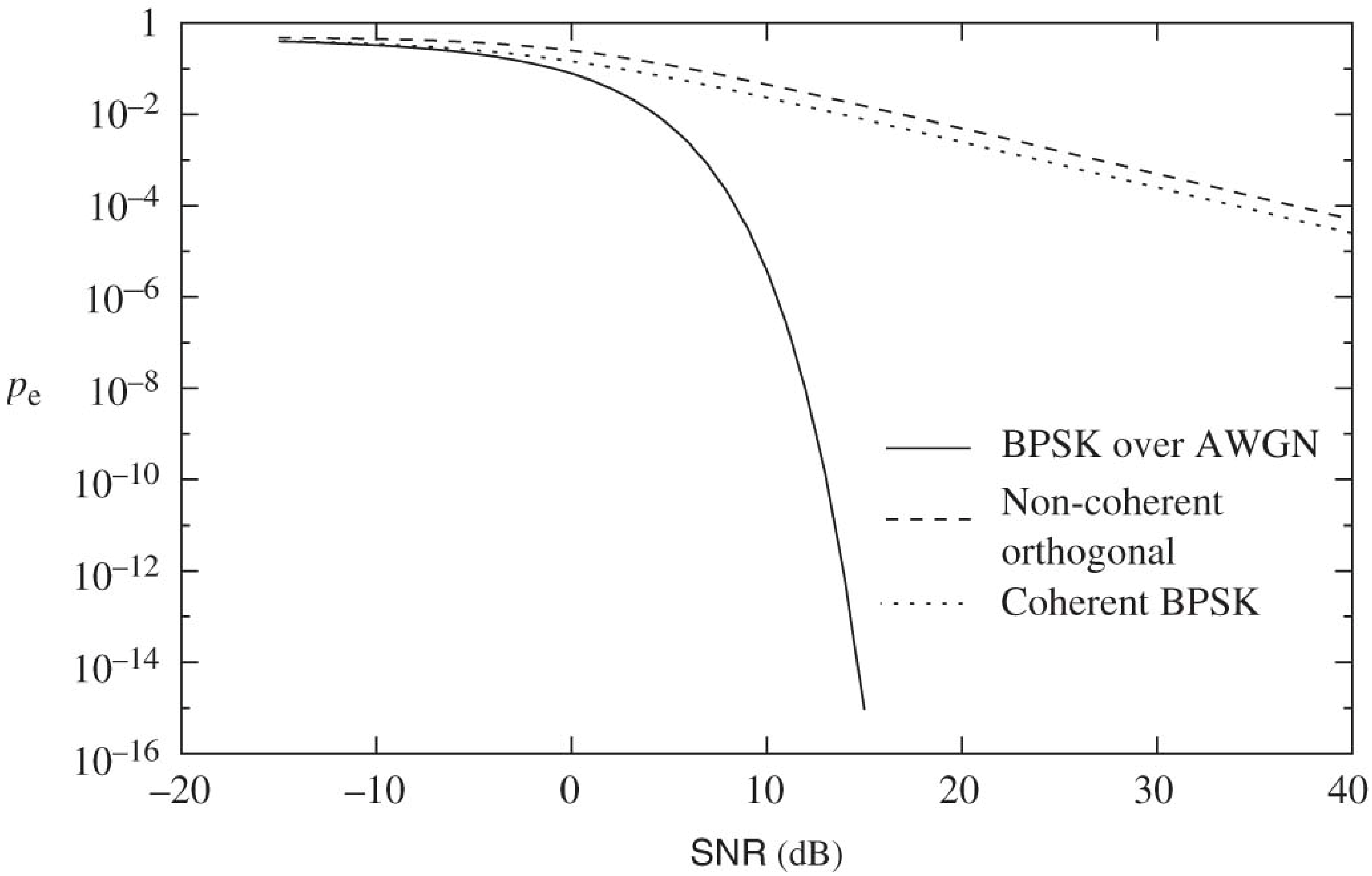
$$p_e = Q\left(\sqrt{2|h|^2 \mathsf{SNR}}\right)$$

Averaged over h,

$$p_e = \frac{1}{2}\left(1 - \sqrt{\frac{\mathsf{SNR}}{1 + \mathsf{SNR}}}\right) \approx \frac{1}{4\mathsf{SNR}}$$

at high SNR.

Rayleigh vs AWGN



Typical Error Event

Conditional on h,

$$p_e = Q\left(\sqrt{2|h|^2\mathsf{SNR}}\right)$$

When $|h|^2 >> \frac{1}{\mathsf{SNR}}$, error probability is very small.

When $|h|^2 < \frac{1}{\mathsf{SNR}}$, error probability is large:

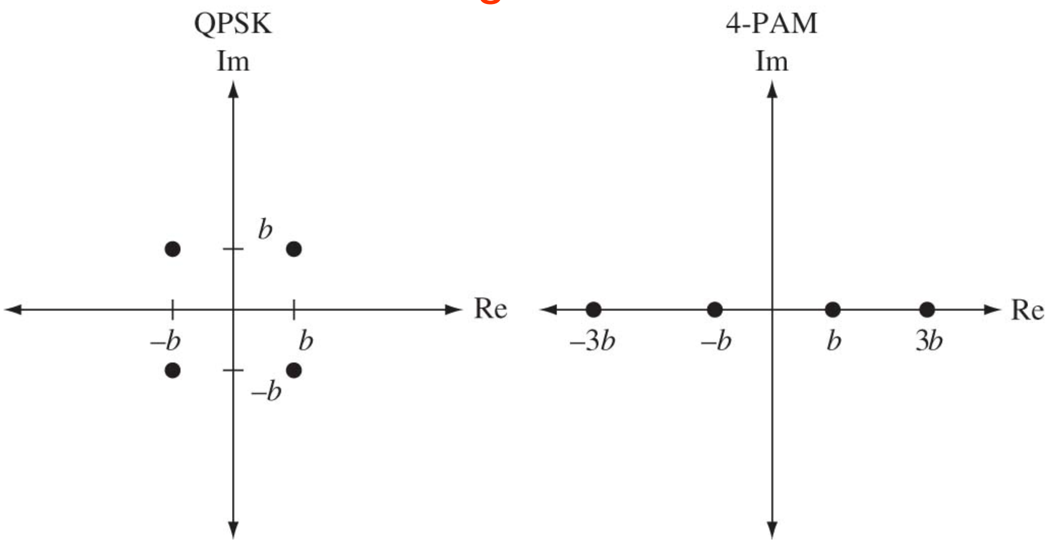
$$p_e \approx P\left(|h|^2 < \frac{1}{\mathsf{SNR}}\right) \approx \frac{1}{\mathsf{SNR}}$$

$$|h|^2 \sim \exp(1).$$

Typical error event is due to channel being in deep fade rather than noise being large.

BPSK, QPSK and 4-PAM

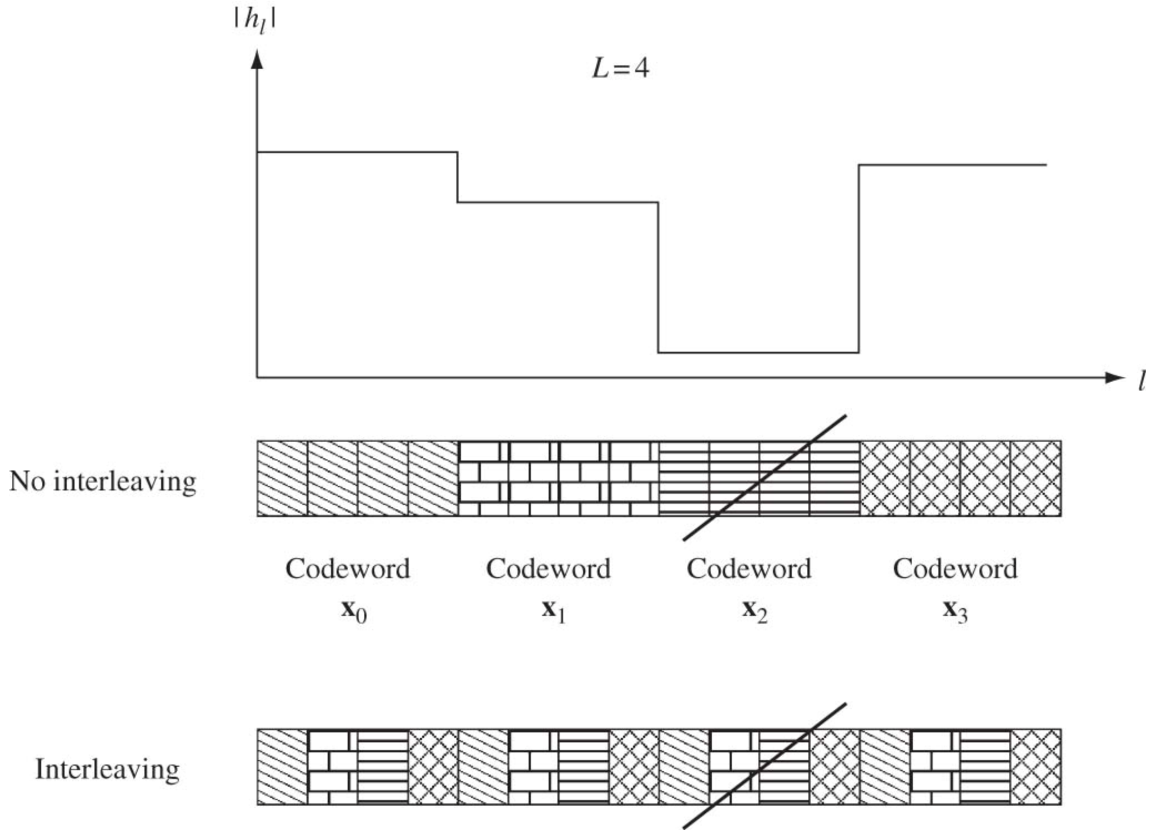
- BPSK uses only the I-phase.The Q-phase is wasted.
- QPSK delivers 2 bits per complex symbol.
- To deliver the same 2 bits, 4-PAM requires 4 dB more transmit power.
- QPSK exploits the available degrees of freedom in the channel better.



- A good communication scheme exploits all the available d.o.f. in the channel.

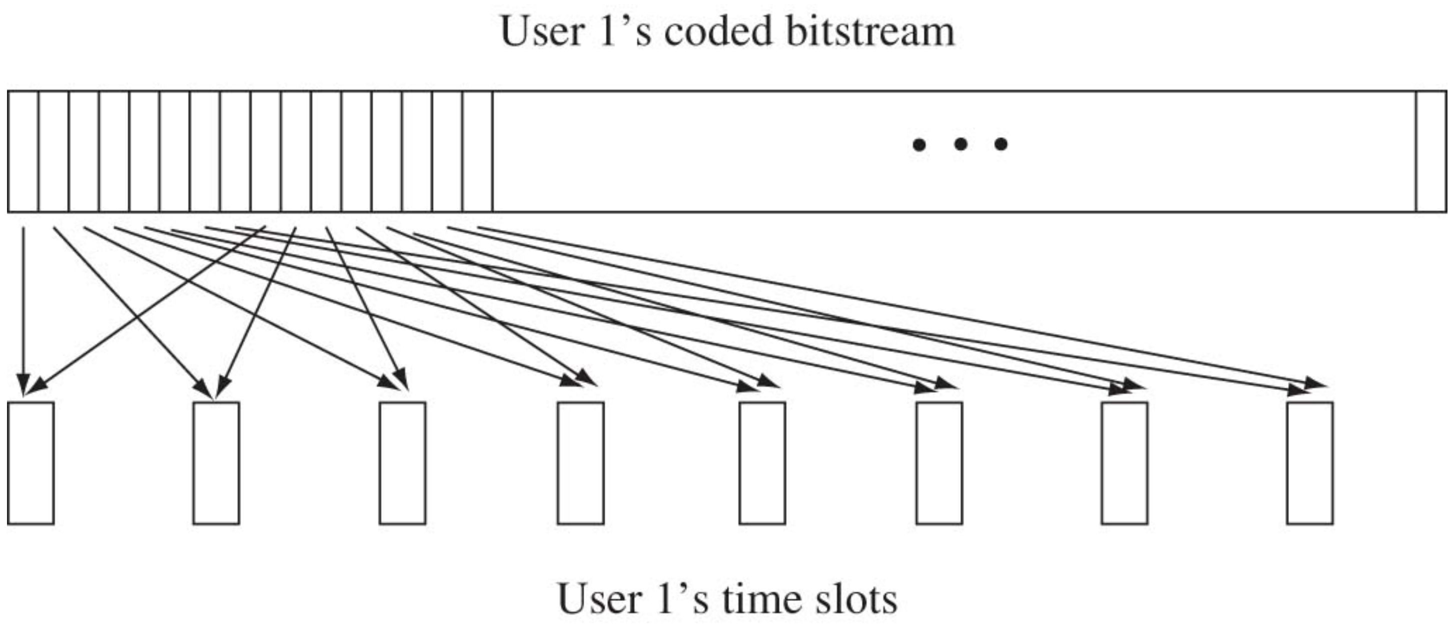
Time Diversity

- Time diversity can be obtained by interleaving and coding over symbols across different coherent time periods.



Coding alone is not sufficient!

Example:GSM



- Amount of time diversity limited by delay constraint and how fast channel varies.
- In GSM, delay constraint is 40 ms (voice).
- To get full diversity of 8, needs v > 30 km/h at f_c = 900 MHz.

Not enough at low speed, then FH between 200 kHz channels.

Simplest Code: Repetition

After interleaving over L coherence time periods,

$$y_\ell = h_\ell x_\ell + w_\ell, \qquad \ell = 1, \ldots, L$$

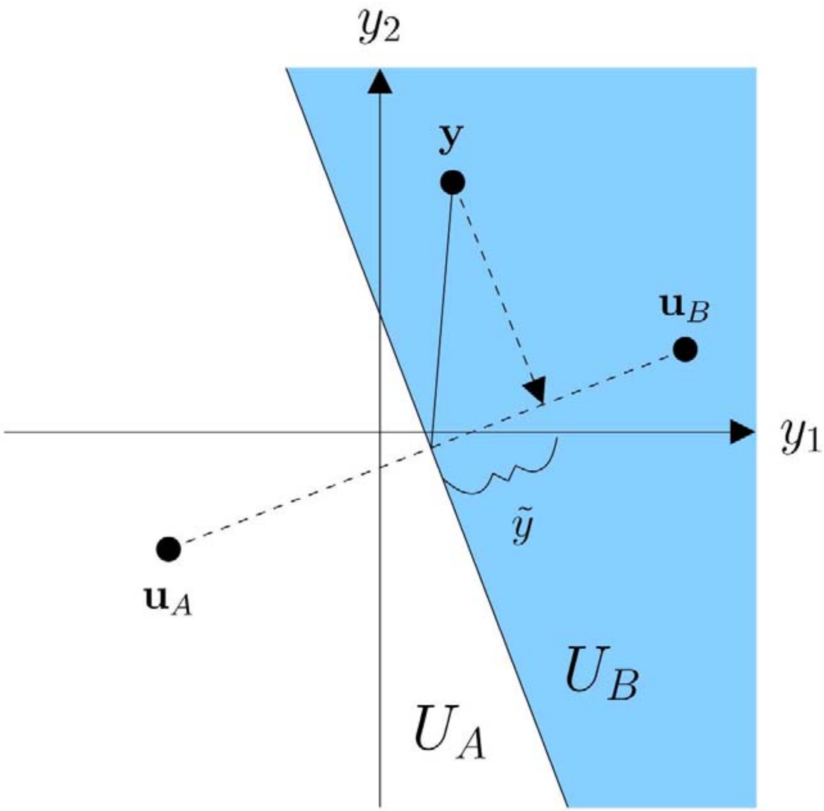
Repetition coding: $x_\ell = x$ for all ℓ.

$$\mathbf{y} = \mathbf{h}x + \mathbf{w}$$

where $\mathbf{y} = [y_1, \ldots, y_L]^t$, $\mathbf{h} = [h_1, \ldots, h_L]^t$, and $\mathbf{w} = [w_1, \ldots, w_L]^t$.

This is classic vector detection in white Gaussian noise.

Geometry



For BPSK $x = \pm a$,

$$\mathbf{u}_A = +a\mathbf{h}, \mathbf{u}_B = -a\mathbf{h}.$$

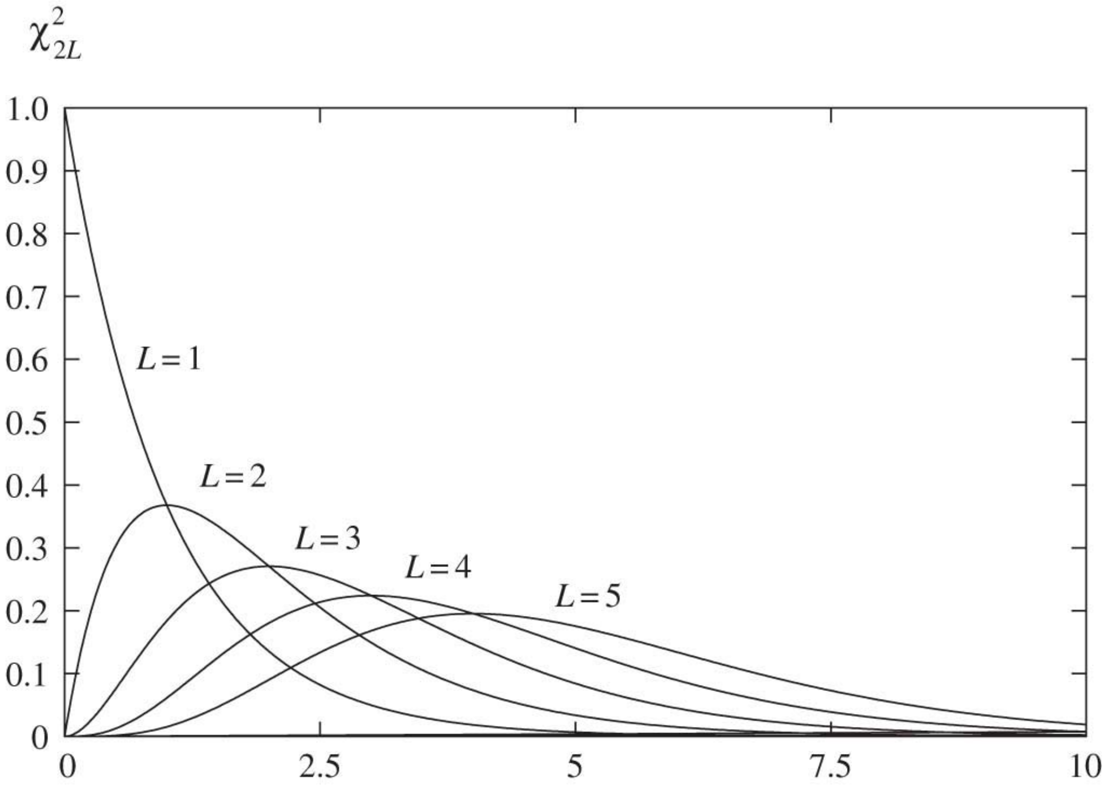
$$\tilde{y} = \frac{\mathbf{h}^*}{\|\mathbf{h}\|}\mathbf{y}$$

Is a sufficient statistic (matched filtering).

Reduces to scalar detection problem:

$$\tilde{y} = \|\mathbf{h}\|x + \tilde{w}$$

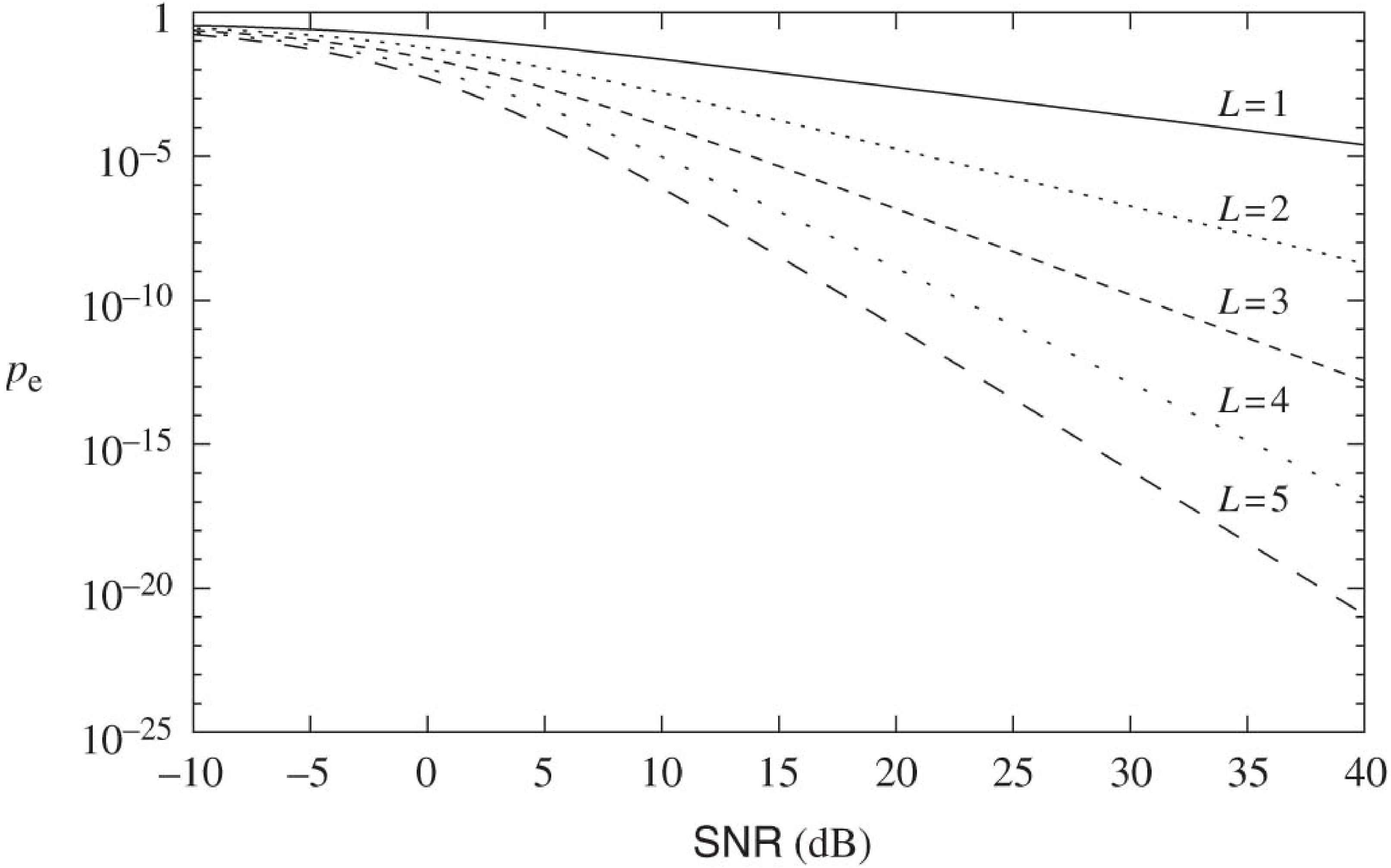
Deep Fades Become Rarer



$$P\left(\|\mathbf{h}\|^2 < \epsilon\right) \approx \frac{1}{L!}\epsilon^L$$

$$p_e \approx P\left(\|\mathbf{h}\|^2 < \frac{1}{\mathsf{SNR}}\right) \approx \frac{1}{L!}\frac{1}{\mathsf{SNR}^L}$$

Performance



Beyond Repetition Coding

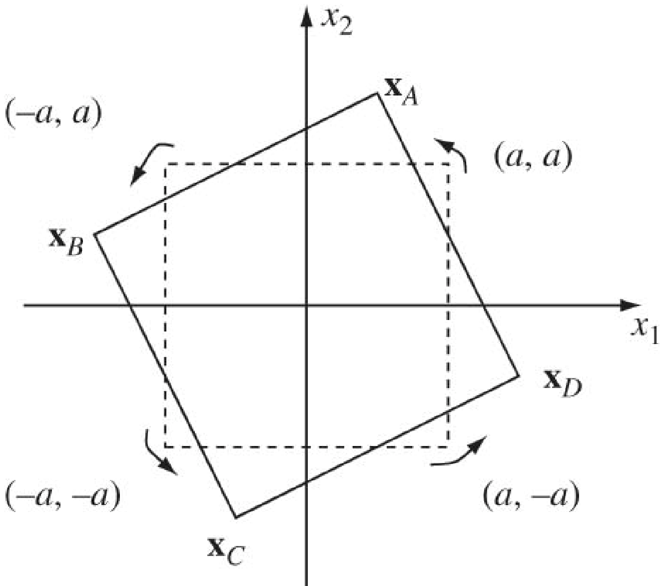
- Repetition coding gets full diversity, but sends only one symbol every L symbol times.
- Does not exploit fully the degrees of freedom in the channel. (analogy: PAM vs QAM)
- How to do better?

Block and convolutional coding in addition to interleaving allow time diversity (diversity gain = minimum Hamming distance). Performance depends on channel statistics (Raleigh).

There are specific coding (based on IT) to define a UNIVERSAL CODE (independent of channle statistics).

Example: Rotation code (L=2)

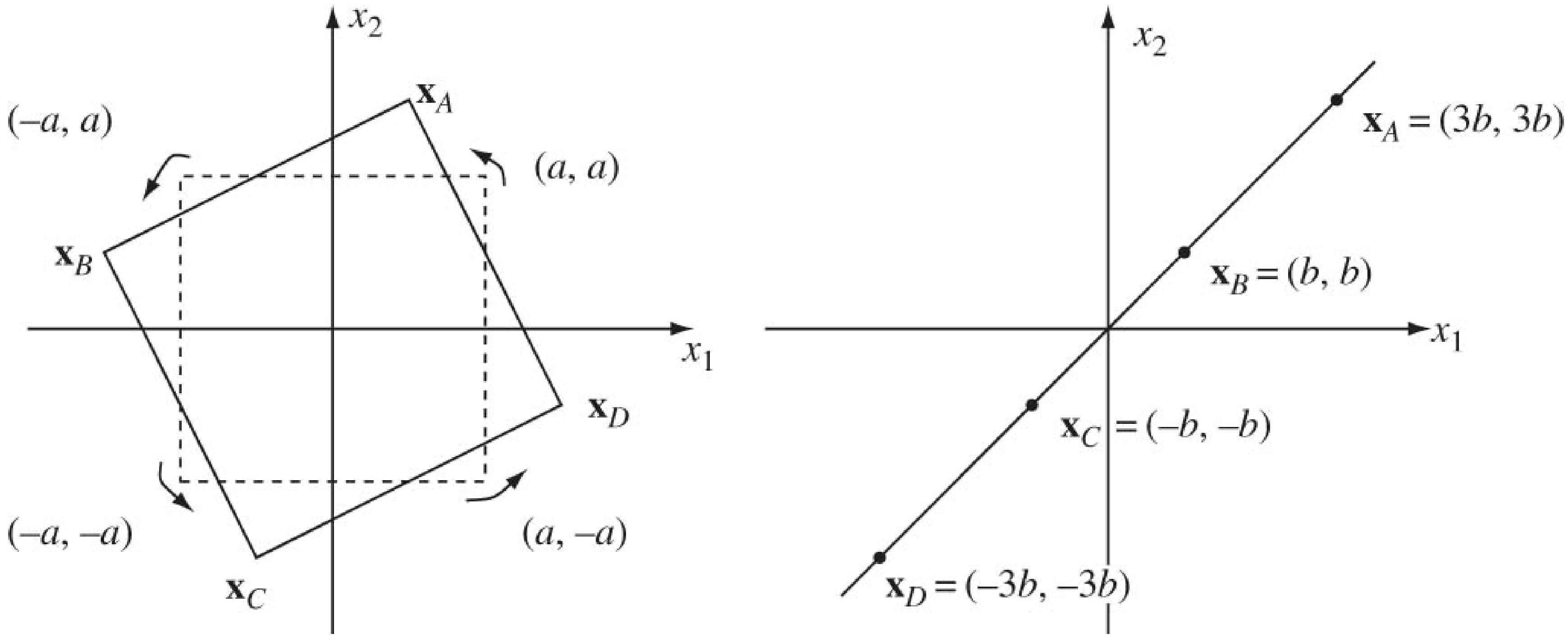
x_1, x_2 are two BPSK symbols before rotation.



$$\mathbb{P}\{\mathbf{x}_A \to \mathbf{x}_B | h_1, h_2\} = Q\left(\frac{\|\mathbf{u}_A - \mathbf{u}_B\|}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\mathsf{SNR}/2 \cdot (|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)}\right)$$

where d_1 and d_2 are the distances between the codewords in the two directions.

Rotation vs Repetition Coding



Rotation code uses the degrees of freedom better!

Product Distance

$$\mathcal{P}\left\{\mathbf{x}_A \to \mathbf{x}_B | h_1, h_2\right\} = Q\left(\sqrt{\frac{\mathsf{SNR}}{2}\left[|d_1|^2|h_1|^2 + |d_2|^2|h_2|^2\right]}\right)$$

$$\begin{aligned}&\mathcal{P}\left\{\mathbf{x}_A \to \mathbf{x}_B\right\}\\ \approx\ & \mathcal{P}\left\{|d_1|^2|h_1|^2 < \frac{1}{\mathsf{SNR}} \quad \& \quad |d_2|^2|h_2|^2 < \frac{1}{\mathsf{SNR}}\right\}\end{aligned}$$

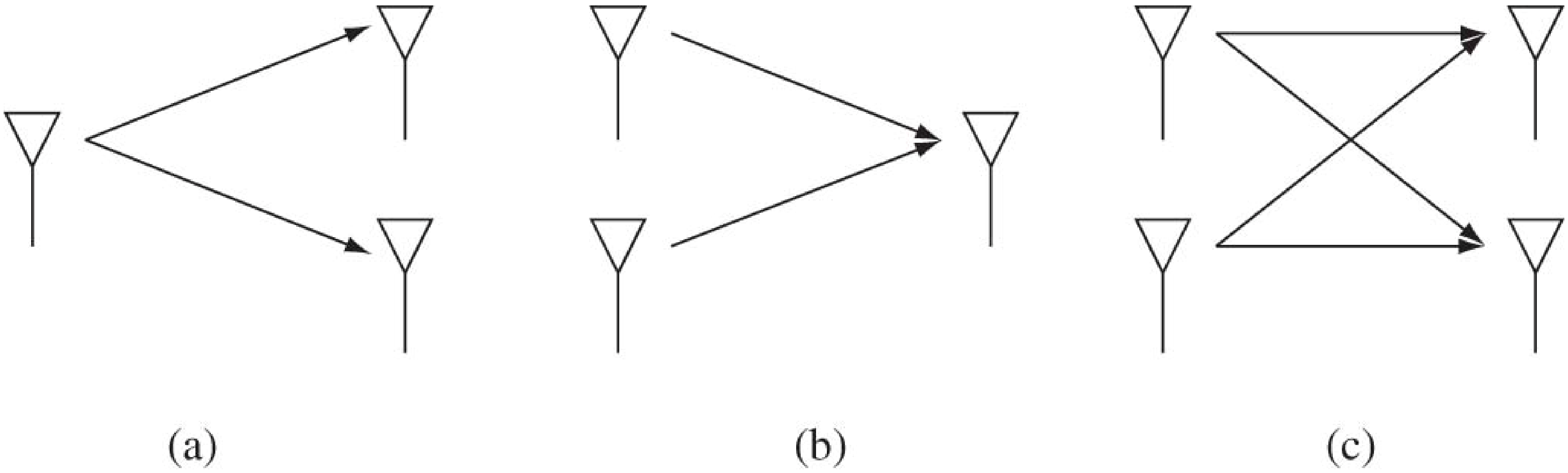
$$\approx \frac{1}{|d_1|^2|d_2|^2}\mathsf{SNR}^{-2}$$

product distance $= |d_1||d_2|$.

Choose the rotation angle to maximize the worst-case product distance to all the other codewords:

$$\theta^* = \frac{1}{2}\tan^{-1} 2.$$

Antenna Diversity



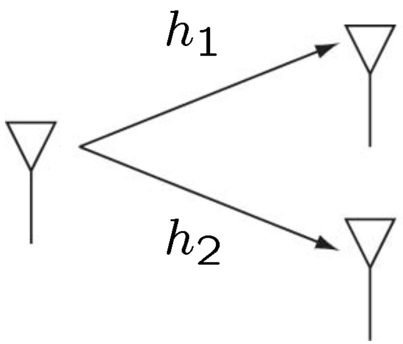
Receive Transmit Both

Diversity depends on antenna distances.

Distances depends on local scattering (for mobile, high: ½ lambda; for base station, low: several lambda) and carrier frequency.

Receive Diversity

$$\mathbf{y} = x\mathbf{h} + \mathbf{w}$$

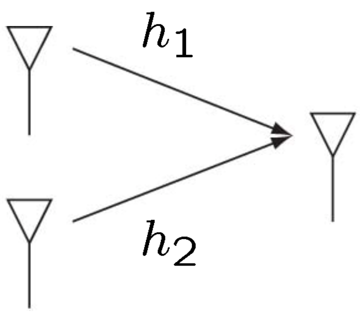


Same as repetition coding in time diversity, except that there is a further power gain.

Optimal reception is via matched filtering (receive beamforming).

Transmit Diversity

$$y = \mathbf{h}^* \mathbf{x} + w$$



If transmitter knows the channel, send:

$$\mathbf{x} = x \frac{\mathbf{h}}{\|\mathbf{h}\|}.$$

maximizes the received SNR by in-phase addition of signals at the receiver (transmit beamforming).

Reduce to scalar channel:

$$y = \|\mathbf{h}\| x + w,$$

same as receive beamforming.

What happens if transmitter does not know the channel?

Space-time Codes

- Transmitting the same symbol simultaneously at the antennas doesn’t work.
- Using the antennas one at a time and sending the same symbol over the different antennas is like repetition coding (bad use of d.o.f).
- More generally, can use any time-diversity code by turning on one antenna at a time.
- Space-time codes are designed specifically for the transmit diversity scenario.

Alamouti Scheme

Over two symbol times:

$$\begin{bmatrix} y[1] & y[2] \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} + \begin{bmatrix} w[1] & w[2] \end{bmatrix}.$$

$$\begin{bmatrix} y[1] \\ y[2]^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w[1] \\ w[2]^* \end{bmatrix}$$

Projecting onto the two columns of the H matrix yields:

$$r_i = \|\mathbf{h}\| u_i + w_i, \qquad i = 1, 2.$$

•double the symbol rate of repetition coding.

•3dB loss of received SNR compared to transmit beamforming.

Space-time Code Design

A space-time code is a set of matrices $\{\mathbf{X}_i\}$.

Full diversity is achieved if all pair-wise differences have full rank.

Coding gain determined by the determinants of $(\mathbf{X}_i - \mathbf{X}_j)(\mathbf{X}_i - \mathbf{X}_i)^*$.

Time-diversity codes have diagonal matrices and the determinant reduces to squared product distances.

Frequency Diversity

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell x[m-\ell] + w[m]$$

- Resolution of multipaths provides diversity.
- Full diversity is achieved by sending one symbol every L symbol times.
- But this is inefficient (like repetition coding).
- Sending symbols more frequently may result in intersymbol interference.
- Challenge is how to mitigate the ISI while extracting the inherent diversity in the frequency-selective channel.

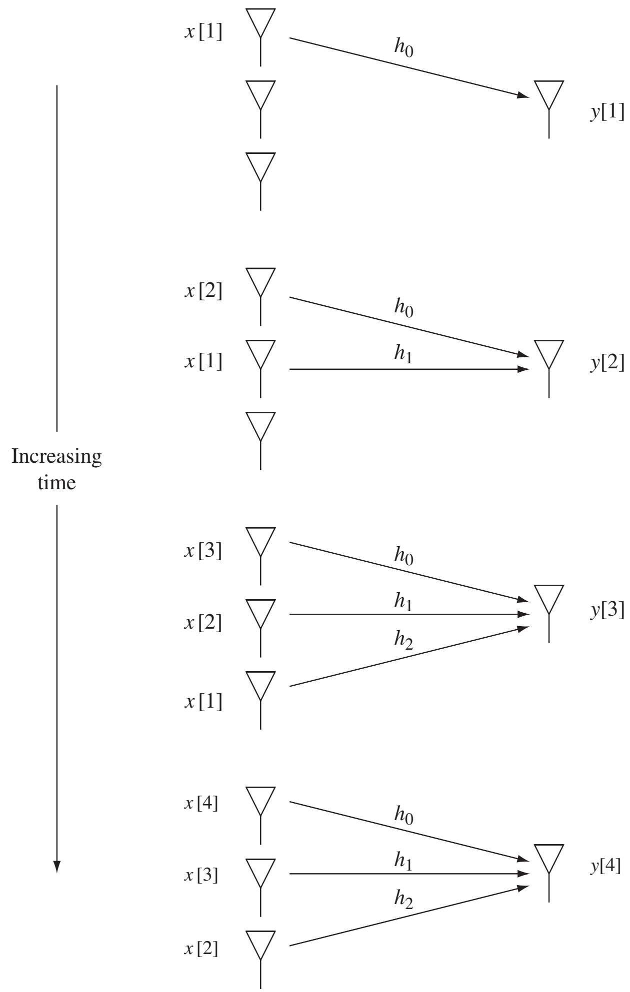
Approaches

- Time-domain equalization (eg. GSM)
- Direct-sequence spread spectrum (eg. IS-95 CDMA)
- Orthogonal frequency-division multiplexing OFDM (eg. 802.11a, Flash-OFDM)

ISI Equalization

- Suppose a sequence of uncoded symbols are transmitted.
- Maximum likelihood sequence detection is performed using the Viterbi algorithm.
- Can full diversity be achieved?

Reduction to Transmit Diversity



MLSD Achieves Full Diversity

Space-time code matrix for input sequence $x[0], \ldots, x[N+L-1]$:

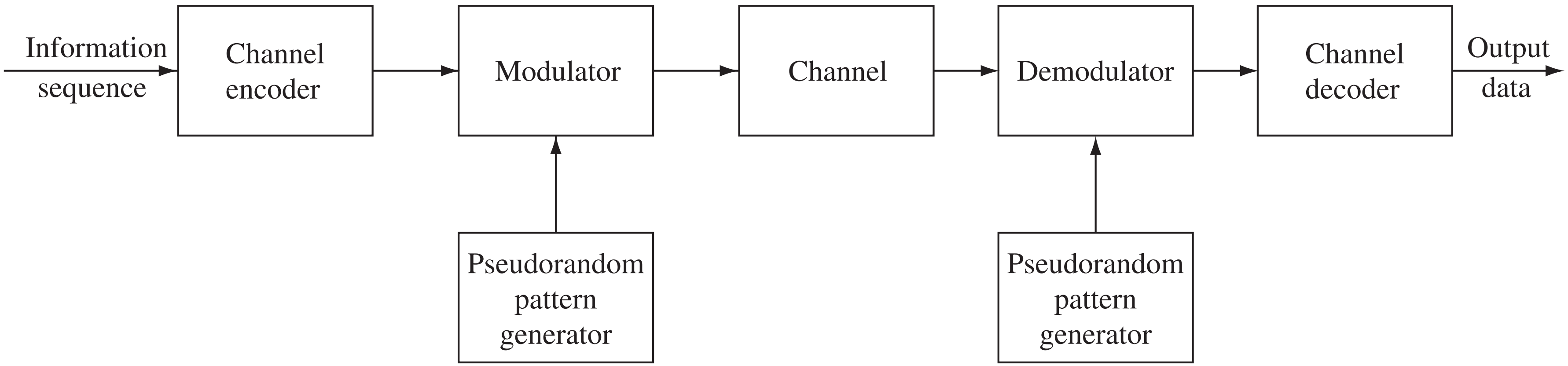
$$\mathbf{X} = \begin{bmatrix} x[1] & x[2] & \cdot & \cdot & \cdot & x[N] & \cdot & \cdot & x[N+L-1] \\ 0 & x[1] & x[2] & \cdot & \cdot & \cdot & x[N] & \cdot & x[N+L-2] \\ 0 & 0 & x[1] & x[2] & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & x[1] & x[2] & \cdot & \cdot & x[N] \end{bmatrix}$$

Difference matrix for two sequences first differing at $m^* \leq N$:

$$\mathbf{X}_A - \mathbf{X}_B = \begin{bmatrix} 0 & \cdot & 0 & x_A[m^*] - x_B[m^*] & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & x_A[m^*] - x_B[m^*] & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & x_A[m^*] - x_B[m^*] & \cdot \end{bmatrix}$$

is full rank.

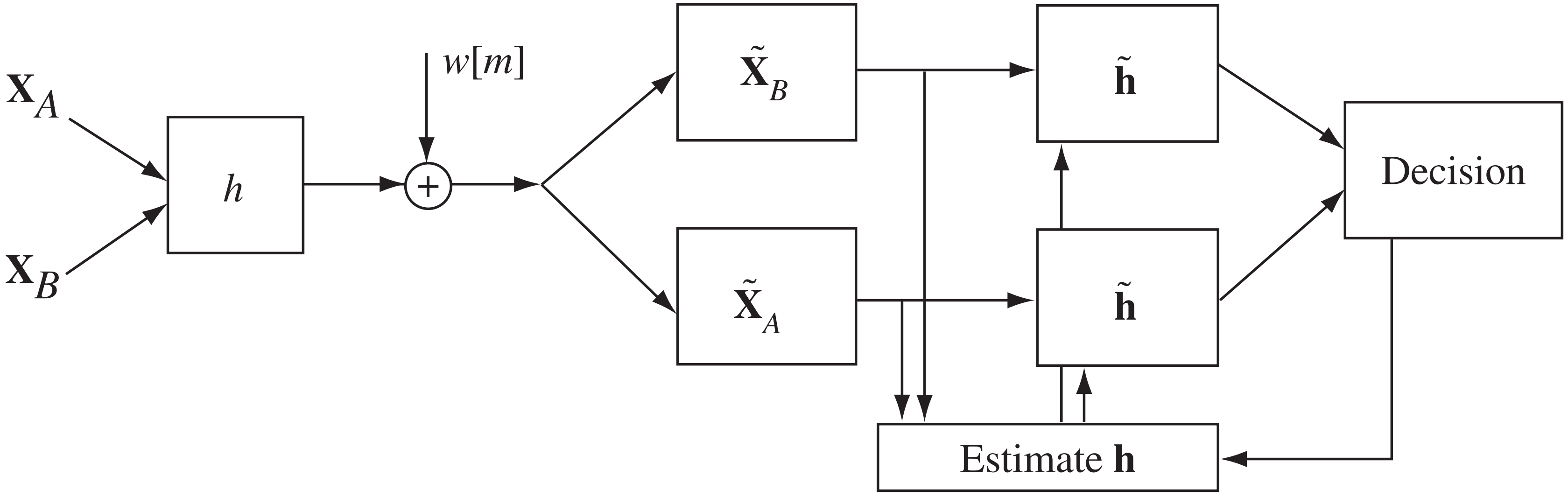
Direct Sequence Spread Spectrum



- Information symbol rate is much lower than chip rate (large processing gain).
- Signal-to-noise ratio per chip is low.
- ISI is not significant compared to interference from other users and matched filtering (Rake) is near-optimal.

Frequency Diversity via Rake

- Considered a simplified situation (uncoded).
- Each information bit is spread into two pseudorandom sequences x_A and x_B ($x_B = -x_A$).



- Each tap of the match filter is a finger of the Rake.

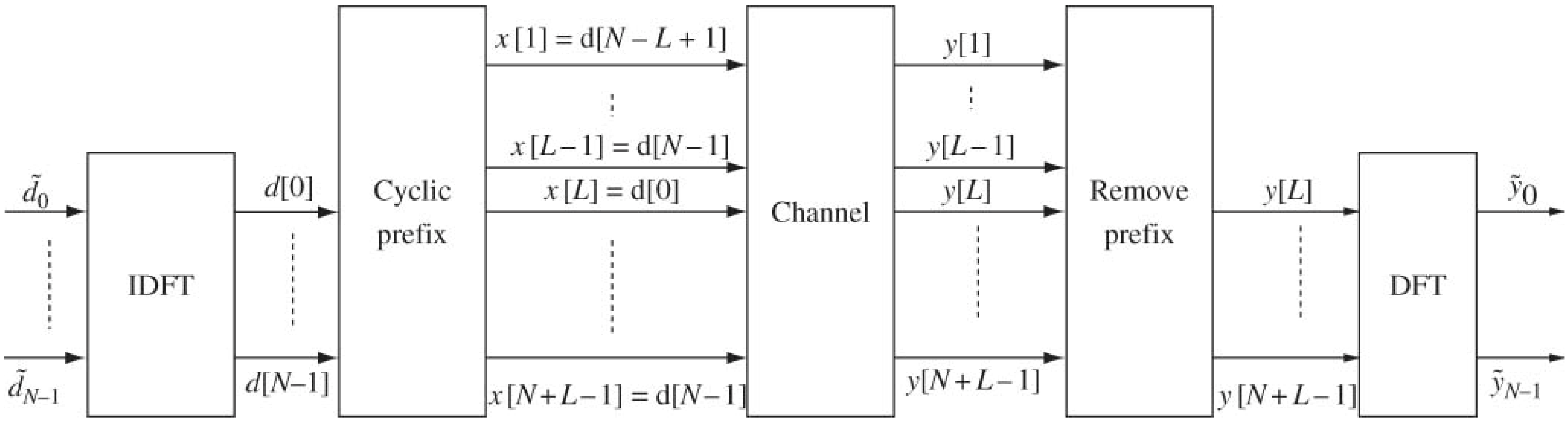
ISI vs Frequency Diversity

- In narrowband systems, ISI is mitigated using a complex receiver.
- In asynchronous CDMA uplink, ISI is there but small compared to interference from other users.
- But ISI is not intrinsic to achieve frequency diversity.
- The transmitter needs to do some work too!

OFDM: Basic Concept

- Most wireless channels are underspread (delay spread << coherence time).
- Can be approximated by a linear time invariant channel over a long time scale.
- Complex sinusoids are the only eigenfunctions of linear time-invariant channels.
- Should signal in the frequency domain and then transform to the time domain.

OFDM



$\tilde{d}_0$
$\tilde{d}_{N-1}$
IDFT
$d[0]$
$d[N-1]$
Cyclic prefix
$x[1] = d[N-L+1]$
$x[L-1] = d[N-1]$
$x[L] = d[0]$
$x[N+L-1] = d[N-1]$
Channel
$y[1]$
$y[L-1]$
$y[L]$
$y[N+L-1]$
Remove prefix
$y[L]$
$y[N+L-1]$
DFT
$\tilde{y}_0$
$\tilde{y}_{N-1}$

OFDM

OFDM transforms the communication problem into the frequency domain:

$$\tilde{y}_n = \tilde{h}_n \tilde{d}_n + \tilde{w}_n, \quad n = 0, \ldots, N_c - 1.$$

a bunch of non-interfering sub-channels, one for each sub-carrier.

$$\tilde{h}_n = H_b\left(\frac{nW}{N_c}\right)$$

Can apply time-diversity techniques.

Cyclic Prefix Overhead

- OFDM overhead

  = length of cyclic prefix / OFDM symbol time
- Cyclic prefix dictated by delay spread.
- OFDM symbol time limited by channel coherence time.
- Equivalently, the inter-carrier spacing should be much larger than the Doppler spread.
- Since most channels are underspread, the overhead can be made small.

Example: Flash OFDM (Flarion)

- Bandwidth = 1.25 MHz
- OFDM symbol = 128 samples = 100 μ s
- Cyclic prefix = 16 samples = 11 μ s delay spread
- 11 % overhead.

Channel Uncertainty

- In fast varying channels, tap gain measurement errors may have an impact on diversity combining performance.
- The impact is particularly significant in channel with many taps each containing a small fraction of the total received energy. (eg. Ultra-wideband channels)
- The impact depends on the modulation scheme.

Summary

- Fading makes wireless channels unreliable.
- Diversity increases reliability and makes the channel more consistent.
- Smart codes yields a coding gain in addition to the diversity gain.
- This viewpoint of the adversity of fading will be challenged and enriched in later parts of the course.